Potential of a Vortex Filament

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Introduction

Problem statement

Given the velocity field

$$v(r) = \frac{1}{4\pi\sqrt{r^2 - (r \cdot n)^2}} \left(\frac{(r + Ln) \cdot n}{|r + Ln|} - \frac{(r - Ln) \cdot n}{|r - Ln|} \right) \frac{r \times n}{|r|}$$

of a vortex filament in direction n and semi–length L.
Compute the according *scalar potential* (assuming inviscid, stationary potential flow).

In another words, we need to find $\Phi : \mathbb{R}^3 \to \mathbb{R}$ s.t. $v = \nabla\Phi$.

Simplifications

- Wlog we assume $n = e_z$ and $L = 1$
- Using cylindrical coordinates r, φ, z

After the calculations:

$$\begin{aligned} v(r,\varphi,z) &= \frac{1}{4\pi\sqrt{r^2+z^2}}\left(\frac{z-1}{\sqrt{r^2+(z-1)^2}} - \frac{z+1}{\sqrt{r^2+(z+1)^2}}\right)e_\varphi \\ &=: -f(r,z)e_\varphi \end{aligned}$$

Analysis of consistency

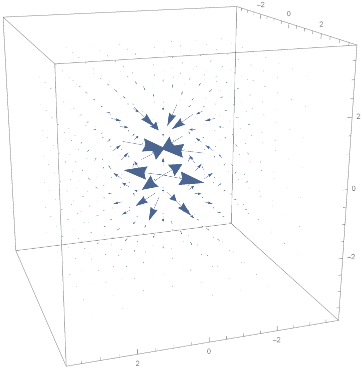

Figure 1: $\nabla \times v$

$$v = \nabla\Phi \implies \nabla \times v = 0.$$

Not equal to 0 obviously!

Vector potential

- Our problem came from flow of incompressible fluid $\implies$

$$\nabla \cdot v = 0$$

- Check: $\nabla \cdot v = -\frac{\partial f(r,z)}{\partial \varphi} = 0$
- Since that exists A s.t.

$$v = \nabla \times A$$

- How to select A?

Selecting an ansatz

- Vector potential is something that generates our vortex
- Since vortex's field has a direction $n = e_z$, it is essential to select

$$A = \begin{pmatrix} 0 \\ 0 \\ F(r,z) \end{pmatrix}$$

$$-f(r,z)e_\varphi = v = \nabla \times A = \nabla \times (F(r,z)e_z) = -\frac{\partial F}{\partial r} e_\varphi$$

The solution

So

$$f(r,z) = \frac{\partial F}{\partial r}$$

From there

$$F(r,z) = \int_0^r f(\rho, z) d\rho + g(z)$$

Assume $g(z) = 0$ and cross fingers...

Exact formula

$$F(r,z) = \frac{i(z-1)EllipticF(i\sinh^{-1}(\frac{r}{|z-1|}), \frac{(z-1)^2}{z^2})}{4\pi|z|} - \frac{i(z+1)EllipticF(i\sinh^{-1}(\frac{r}{|z|}), \frac{z^2}{(z+1)^2})}{4\pi|z+1|}$$

$$EllipticF(\varphi, k) = \int_0^{\varphi} (\frac{1}{\sqrt{1 - ksin^2\psi}}) d\psi$$

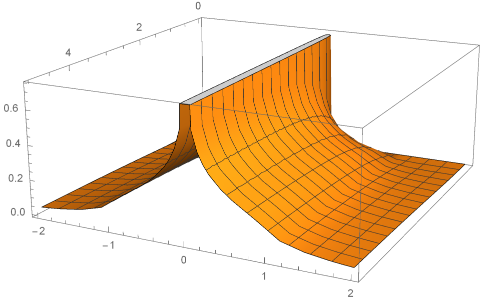

Figure 2: F(r, z)

Further work

We found the solution!
But it is also interesting to find the potential A' s.t. $\nabla \cdot A' = 0$.
Assume

$$A' = A + \nabla\Phi$$

This implies

$$\nabla \times A' = \nabla \times A = v$$

Also

$$\nabla \cdot A' = \nabla \cdot A + \Delta\Phi = 0$$

So we have Poisson equation for Φ.
Because of the form of $\nabla \cdot A$ it is *not possible* to solve it analytically.

Remarks about fluid motion

Navier-Stokes equation for inviscid stationary potential flow:

$$\rho(v \cdot \nabla)v = -\nabla p + \rho F$$

Recalling $v = f(r,z)e_\varphi$, you can show that

$$(v \cdot \nabla)v = \begin{pmatrix} -\frac{f^2(r,z)}{r} \\ 0 \\ 0 \end{pmatrix}$$

Since that

$$-\rho\frac{f^2(r,z)}{r} = -\frac{\partial p}{\partial r} + \rho F_r \quad \frac{\partial}{\partial z}$$

$$0 = -\frac{\partial p}{\partial \varphi} + \rho F_\varphi$$

$$0 = -\frac{\partial p}{\partial z} + \rho F_z \quad \frac{\partial}{\partial r}$$

Remarks about fluid motion

So, we have

$$\frac{2f}{r}\frac{\partial f}{\partial z} = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \neq 0$$

But since F is a potential force

$$\nabla \times F = \begin{pmatrix} \dots \\ \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \\ \dots \end{pmatrix} = 0!$$

Possible reasons:

- Non potential force
- Influence of boundary conditions

Conclusion

Obtained results:

- There is no *scalar potential* solution
- Analytical solution for *vector potential* was provided

Further research:

- Find the divergence-free solution
- Investigation of fluid dynamics equation